RESEARCH STATEMENT

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I work in geometric representation theory, a field which is concerned with understanding the symmetries of various algebraic varieties, and connecting them to mathematical physics, symplectic geometry and combinatorics.

More specifically, my research focuses on two algebras: the quantum toroidal algebra and K-theoretical Hall algebra. My study includes many different aspects, including the categorification, the Feigin-Odesskiil wheel conditions and the incarnation with moduli space of sheaves on surfaces, Calabi-Yau 3-folds and Kuznetsov components.

My research papers are published in [23](published by IMRN, doi number /imrn/rnaa123), [21], and [22].

1. A Categofical Quantum Toroidal Action on Hilbert Schemes

The quantum toroidal algebra $U_{q_1,q_2}(\ddot{gl}_1)$ is an affinization of the quantum Heisenberg algebra, studied by Ding-Iohara-Miki [3, 13], and has been realized in several contexts:

- the elliptic Hall algebra in [1, 18],
- the double shuffle algebra in [4, 6, 16],
- the trace of the deformed Khovanov Heisenberg category in [2](when $q_1 = q_2$).

Given a smooth quasi-projective surface S over $k = \mathbb{C}$, let

$$\mathcal{M} = \bigsqcup_{n=0}^{\infty} S^{[n]}$$

be the Hilbert schemes of points on S. Schiffmann-Vasserot [20], Feigin-Tsymbaliuk [7] and Neguţ [17] constructed the $U_{q_1,q_2}(\ddot{gl}_1)$ action on the Grothendieck group of $\mathcal{M}$. It generalizes the action of

- the Heisenberg algebra (Nakajima [14] and Grojnowski [8])
- the W algebra (Li-Qin-Wang [11])

on the cohomology of Hilbert schemes. There are already some applications of this action in algebraic geometry, like the Beauville-Voison conjecture for the Hilbert schemes of points on $K3$ surfaces [12].

In [22], we categorify the above quantum toroidal algebra action. We prove that

Theorem 1.1. *Consider the correspondences $e_k, f_k : D^b(\mathcal{M}) \to D^b(\mathcal{M} \times S)$ induced from:*

$$e_k := \mathcal{L}^k \mathcal{O}_{S^{[n,n+1]}} \in D^b(S^{[n]} \times S^{[n+1]} \times S)$$
$$f_k := \mathcal{L}^{k-1} \mathcal{O}_{S^{[n,n+1]}}[1] \in D^b(S^{[n+1]} \times S^{[n]} \times S).$$

where $S^{[n,n+1]}$ *is the nested Hilbert scheme*

$$S^{[n,n+1]} := \{(\mathcal{I}_n, \mathcal{I}_{n+1}, x) \in S^{[n]} \times S^{[n+1]} \times S | \mathcal{I}_{n+1} \subset \mathcal{I}_n, \mathcal{I}_n/\mathcal{I}_{n+1} = k_x\}$$

and the line bundle $\mathcal{L}$ *on* $S^{[n,n+1]}$ *has fibers equal to* $\mathcal{I}_n/\mathcal{I}_{n+1}$*. Then*

(1) *For every two integers* m *and* r*, there exists natural transformations*

$$\begin{cases} f_r e_{m-r} \to e_{m-r} f_r & \text{if } m > 0 \\ e_{m-r} f_r \to f_r e_{m-r} & \text{if } m < 0 \\ f_r e_{-r} = e_r f_{-r} \oplus \mathcal{O}_\Delta[1] \end{cases} \tag{1.1}$$

where Δ *is the diagonal of* $\mathcal{M} \times \mathcal{M} \times S \times S$.

(2) *When* $m \neq 0$*, the cone of the natural transformations in* (1.1) *has a filtration with associated graded object*

$$\begin{cases} \bigoplus_{k=0}^{m-1} R\Delta_*(h^+_{m,k}) & \text{if } m > 0 \\ \bigoplus_{k=m+1}^{0} R\Delta_*(h^-_{m,k}) & \text{if } m < 0 \end{cases}$$

where $h^+_{m,k}, h^-_{m,k} \in D^b(\mathcal{M} \times S)$ *are combinations of wedge and symmetric product of complexes of universal sheaves on* $\mathcal{M} \times S$.

(3) *At the level of Grothendieck groups, we have the formula:*

$$(1 - [\omega_S]) \sum_{k=0}^{m-1} [h^+_{m,k}] = h^+_m \qquad m > 0$$

$$(1 - [\omega_S]) \sum_{k=m+1}^{0} [h^-_{m,k}] = h^-_{-m} \qquad m < 0$$

where ω_S *is the canonical line bundle of* S *and* $h^\pm_m$ *is defined in* [15] *.*

Note that the non-triviality of the extension is a feature of the derived category statement, which is not visible at the level of Grothendieck groups. We also provide a precise extension formula.

The positive part of the quantum toroidal algebra action was already categorified in [17]. Theorem 1.1 categorifies the commutation of the positive part and the negative part, and thus accounts for the action of the whole quantum toroidal algebra.

2. K-theoreitc Hall algebra of a surface

Schiffmann-Vasserot [19] and Kontsevich-Soibelman [10] initiated the study of Cohomological/K-theoretic Hall algebra. The K-theoretic Hall algebra of a surface generalizes Nakajima's and Grojnowski's work in the following two ways:

Hilbert schemes $\to$moduli space of sheaves/instantons
homology groups $\to$algebraic K-theory

In [20], Schiffmann-Vasserot laid out the following philosophy for studying K-theoretic Hall algebras. Consider the moduli stack

$$Coh = \bigsqcup_{n \in \mathbb{Z}^{\geq 0}} Coh_n$$

where Coh_n is the moduli stack of dimension 0 length n coherent sheaves over S. For any $n, m \in \mathbb{Z}^{\geq 0}$, we consider the moduli stack $Coh^{ext}_{n,m}$ which consists of short exact sequences

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n+m} \to \mathcal{E}_m \to 0$$

where $\mathcal{E}_n \in Coh_n$, $\mathcal{E}_m \in Coh_m$ and the following diagram, which effectively generalizes the notion of Hecke correspondences to surfaces:

$$\begin{array}{ccc} & \{0 \to \mathcal{E}_n \to \mathcal{E}_{n+m} \to \mathcal{E}_m \to 0\} \in Coh^{ext}_{n,m} & \\ \swarrow^{p_{n,m}} & & \searrow^{q_{n,m}} \\ (\mathcal{E}_n, \mathcal{E}_m) \in Coh_n \times Coh_m & & \mathcal{E}_{n+m} \in Coh_{n+m}, \end{array}$$

The expectation of [20] is that

$$q_{n,m_*} \circ p^*_{n,m} : K(Coh_n) \otimes K(Coh_m) \to K(Coh_{n+m})$$

induces an associative algebra structure on

$$K(Coh) = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} K(Coh_n),$$

which is called the K-theoretic Hall algebra of S. $K(Coh)$ has a natural action on $K(\mathcal{M})$ for various moduli spaces of coherent sheaves/instantons $\mathcal{M}$.

The main technical difficulty is correctly defining $p^*_{n,m}$, since $p_{n,m}$ is not a local complete intersection morphism. In [19], Schiffmann-Vasserot gave a definition for $S = \mathbb{A}^2$, but the general case was lacking. I gave a definition which works for general S in [23](which was soon followed by an independent and philosophically similar paper Kapranov-Vasserot in [9]), and prove

Theorem 2.1. *$q_{n,m_*} \circ p^*_{n,m}$ induces an associative algebra structure $(K(Coh), *^{K(Coh)})$.*

3. Shuffle Presentation and the Feigin-Odesskii Wheel Conditions

The K-theoretic Hall algebra of a surface includes interesting algebraic structures, like the positive part of Heisenberg algebra and the positive part of Ding-Iohara-Miki algebra ([20],[7]). The shuffle algebra provides us an approach to handle the whole structure of $K(Coh)$ through an algebraic homomorphism

$$K(Coh) \to \bigoplus_{n=0}^{\infty} K(S^n)(z_1, \dots, z_n)^{W_n}$$

For any quasi-projective surface S and a non-negative integer n,

$$Coh_n = [Quot^{\circ}_n / GL_n]$$

where $Quot^{\circ}_n$ is the moduli spaces of quotients of coherent sheaves

$$\phi : \mathcal{O}^n \twoheadrightarrow \mathcal{E}_n,$$

where $\mathcal{E}_n$ is a dimension 0 and length n coherent sheaf, and $H^0(\phi) : k^n \to H^0(\mathcal{E}_n)$ is an isomorphism. Hence

$$K(Coh_n) = K^{GL_n}(Quot^{\circ}_n) = K^{T_n}(Quot^{\circ}_n)^{W_n},$$

where W_n is the Weyl group of $GL(n)$. The maximal torus T_n acts on $Quot_n^\circ$ and the fixed locus is S^n. The Thomason localization theorem implies an homomorphism of abelian groups:

$$\tau = \bigoplus_{n=0}^{\infty} \tau_n : K(Coh) \to \bigoplus_{n=0}^{\infty} K(S^n)(z_1, \cdots, z_n)^{W_n}$$

which can be upgraded to an algebra homomorphism by using the result of Theorem 2.1.

Theorem 3.1. *Let*

$$Sh := \bigoplus_{n=0}^{\infty} K(S^n)(z_1, \cdots, z_n)^{W_n},$$

with the associative product $*^{Sh}$*:*

$$R(z_1, ..., z_n) *^{Sh} R'(z_1, ..., z_m) = \tag{3.1}$$

$$= Sym\left[(R \boxtimes 1^{\boxtimes m})(z_1, ..., z_n)(1^{\boxtimes n} \boxtimes R')(z_{n+1}, ..., z_{n+m}) \prod_{i=1}^{n} \prod_{j=n+1}^{n+m} \zeta_{ij}^S \left(\frac{z_j}{z_i}\right)\right]$$

where:

$$\zeta_{ij}^S(x) = \frac{[\wedge^\bullet(x \cdot \mathcal{O}_{\Delta_{ij}})]}{(1-x)(1-\frac{1}{x})} \in K_{S^{n+m}}(x)$$

and $\mathcal{F} \boxtimes \mathcal{G} = pr_n^*(\mathcal{F}) \otimes pr_m^*(\mathcal{G})$ *for* $\mathcal{F} \in K(S^n)$, $\mathcal{G} \in K(S^m)$. *Here* pr_n, pr_m *are the respective projection maps from* $S^n \times S^m$ *to* S^n *and* S^m. *Then the Thomason localization theorem induces an algebra homomorphism*

$$\tau : (K(Coh), *^{K(Coh)}) \to (Sh, *^{Sh}).$$

Moreover, in [21], I gave a geometric explanation of the "wheel conditions" and the "pole conditions", which is first studied by Feigin-Odesskii [5], and generalized them to any smooth surface S.

Theorem 3.2. *Let* S *be a smooth quasi-projective surface, there are explicit vector bundles* $\mathcal{W}_{ij}$ *on* S^n, *such that for any element* $[\mathcal{F}]$ *in the image of* τ_n, *we have*

1. *(Pole Conditions)* $(\prod_{i\neq j}[\wedge^\bullet \frac{z_j}{z_i}\mathcal{W}_{ij}]) \otimes [\mathcal{F}] \in K(S^n)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$
2. *(Wheel Conditions) For any three different numbers* $i, j, k \in \{1, \dots, n\}$, *let*

$$\Delta_{ijk} = \{(s_1, \dots, s_n) \in S^n | s_i = s_j = s_k\}$$

and $pr : \Delta_{ijk} \to S$ *be the morphism which maps* $(s_1, \dots, s_n)$ *to* s_i. *Then* $((\prod_{i\neq j}[\wedge^\bullet \frac{z_j}{z_i}\mathcal{W}_{ij}]) \otimes [\mathcal{F}])|_{\Delta_{ijk}}$ *is in the ideal generated by* $(1 - \frac{z_i}{z_k} pr^*\omega_S)$, *and* $(1 - q_1\frac{z_j}{z_k})(1 - q_2\frac{z_j}{z_k})$, *where* q_1, q_2 *are the two Chern roots of* $pr^*\Omega_S$.

4. Some Ongoing Projects

Following the strategy of [22], I am preparing a paper about the Heisenberg categorification of Hilbert schemes of K3 surfaces.

Together with Xiaolei Zhao and Huachen Chen, we are also studying the representation structure of the Kusznetsov components.

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