

A (Weak) Quantum Toroidal Categorification On Hilbert Schemes

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Overview

- Schiffmann-Vasserot, Feigin-Tsybaliuk constructed $U_{q_1, q_2}(\mathfrak{gl}_1)$ action on the Grothendieck groups of Hilbert schemes of points on surfaces.

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- It generalized the Heisenberg algebra action on the cohomology by Nakajima and Grojnowski.
- In this talk, we will categorify the above $U_{q_1, q_2}(\ddot{gl}_1)$ action.

Quantum Toroidal Algebra Action on The Grothendieck Groups of Hilbert Schemes

Hilbert Schemes and Cohomology

Given a quasi-projective smooth surface S over $k = \mathbb{C}$, we consider $S^{[n]}$ the Hilbert scheme of n points on S , and let

$$\mathcal{M} := \bigsqcup_{n=0}^{\infty} S^{[n]}$$

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Theorem (Nakajima, Grojnowski)

The homology group $H_(M)$ is a irreducible highest weight representation as a representation of the Heisenberg superalgebra associated with $H^*(X)$, where the highest weight vector is the generator of $H_0(X^{[0]}) \cong \mathbb{Q}$.*

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Quantum Toroidal Algebra $U_{q_1, q_2}(\ddot{\mathfrak{gl}}_1)$

- $U_{q_1, q_2}(\ddot{\mathfrak{gl}}_1)$ is an affinization of the q -Heisenberg algebra.
- The study of $U_{q_1, q_2}(\ddot{\mathfrak{gl}}_1)$ started from many different origins in algebraic geometry, representation theory and mathematical physics.
- Let $\mathbb{K} = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]_{([1], [2], [3], \dots)}^{\text{Sym}}$ where “Sym” means symmetric in q_1 and q_2 and let $q = q_1 q_2$. Then $U_{q_1, q_2}(\ddot{\mathfrak{gl}}_1)$ is the \mathbb{K} -algebra with generators

$$\{E_k, F_k, H_l^{\pm}\}_{k \in \mathbb{Z}, l \in \mathbb{N}}$$

modulo the following relations:

Quantum Toroidal Algebra $U_{q_1, q_2}(\mathfrak{gl}_1)$

$$(z - wq_1)(z - wq_2)(z - \frac{w}{q})E(z)E(w) = (z - \frac{w}{q_1})(z - \frac{w}{q_2})(z - wq)E(w)E(z) \quad (1)$$

$$(z - wq_1)(z - wq_2)(z - \frac{w}{q})E(z)H^\pm(w) = (z - \frac{w}{q_1})(z - \frac{w}{q_2})(z - wq)H^\pm(w)E(z) \quad (2)$$

$$[[E_{k+1}, E_{k-1}], E_k] = 0 \quad \forall k \in \mathbb{Z} \quad (3)$$

together with the opposite relations for $F(z)$ instead of $E(z)$, and:

$$[E(z), F(w)] = \delta(\frac{z}{w})(1 - q_1)(1 - q_2)(\frac{H^+(z) - H^-(w)}{1 - q}) \quad (4)$$

where

$$E(z) = \sum_{k \in \mathbb{Z}} \frac{E_k}{z^k}, \quad F(z) = \sum_{k \in \mathbb{Z}} \frac{F_k}{z^k}, \quad H^\pm(z) = \sum_{l \in \mathbb{N} \cup \{0\}} \frac{H_l^\pm}{z^{\pm l}} \quad (5)$$

where

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{Q}\{\{z\}\}.$$

We will set $H_0^+ = q$ and $H_0^- = 1$.

Hilbert Schemes and Grothendieck Groups

- Let $S^{[n,n+1]}$ be the nested Hilbert scheme parameterized by $\{(\mathcal{I}_n, \mathcal{I}_{n+1}, x) \in S^{[n]} \times S^{[n+1]} \times S \mid \mathcal{I}_{n+1} \subset \mathcal{I}_n, \mathcal{I}_n/\mathcal{I}_{n+1} = k_x\}$.

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$$e_k := [\mathcal{L}^k \mathcal{O}_{S^{[n,n+1]}}], \quad f_k = [\mathcal{L}^{k-1} \mathcal{O}_{S^{[n,n+1]}}]$$

could be regarded as operators: $K(\mathcal{M}) \rightarrow K(\mathcal{M} \times S)$ through the K -theoretic correspondences.

Hilbert Schemes and Grothendieck Groups

Theorem (Schiffmann-Vasserot, Feigin-Tsybaliuk)

- *There exists $h_m \in K(\mathcal{M} \times S)$ which is a combination of symmetric product and wedge product of the universal ideal sheaf on $\mathcal{M} \times S$ such that*

$$[e_k, f_l] = \Delta_* \left(\frac{h_{k+l}}{1-q} \right)$$

where $\Delta : \mathcal{M} \times S \rightarrow \mathcal{M} \times \mathcal{M} \times S \times S$ is the diagonal embedding and $q = [\omega_S]$ is the canonical line bundle of S .

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- e_k, f_k, h_k satisfy the relations in $U_{q_1, q_2}(\check{g}l_1)$.

Some Applications

- It also acts on the Grothendieck group of higher rank stable sheaves, and factors through the deformed W -algebra. It leads to a proof of AGT correspondences for $U(r)$ gauge theory with matter. (Neguț)

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- It also acts on the Grothendieck group of higher rank stable sheaves, and factors through the deformed W -algebra. It leads to a proof of AGT correspondences for $U(r)$ gauge theory with matter. (Neguț)
- It induces a W -algebra action on the Chow groups of Hilbert schemes of points on surfaces which is studied by Li-Qin-Wang in the homology theory, which induced the Beauville conjecture on the Hilbert schemes of $K3$ surfaces (Maulik-Neguț).

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(Weak) Categorification of $U_{q_1, q_2}(\ddot{\mathfrak{gl}}_1)$

- The action of the positive part was also categorified by Negut
- A monoidal categorification of the positive part is given by Porta-Sala through the Categorified Hall algebra.
- We will categorify the commutator of the positive and the negative part:

$$[e_k, f_l] = \Delta_* \frac{h_{k+l}}{1-q}$$

by constructing natural transformations in derived categories explicitly.

The Main Theorem(Y. Zhao)

- For every two integers m and r , there exists natural transformations

$$\begin{cases} f_r e_{m-r} \rightarrow e_{m-r} f_r & \text{if } m > 0 \\ e_{m-r} f_r \rightarrow f_r e_{m-r} & \text{if } m < 0 \\ f_r e_{-r} = e_r f_{-r} \oplus \mathcal{O}_{\Delta}[1]. \end{cases} \quad (6)$$

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- When $m \neq 0$, the cone of above natural transformations has a filtration with associated graded object

$$\begin{cases} \bigoplus_{k=0}^{m-1} R\Delta_*(h_{m,k}^+) & \text{if } m > 0 \\ 0 & \\ \bigoplus_{k=m+1} R\Delta_*(h_{m,k}^-) & \text{if } m < 0 \end{cases}$$

where $h_{m,k}^+, h_{m,k}^- \in D^b(\mathcal{M} \times S)$ are combinations of wedge and symmetric products of universal sheaves on $\mathcal{M} \times S$.

The main theorem

- At the level of Grothendieck groups, we have the formula:

$$(1 - [\omega_S]) \sum_{k=0}^{m-1} [h_{m,k}^+] = h_m^+ \quad m > 0$$

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- The extensions between $h_{m,k}^\pm$ are non-trivial and given by a explicit formula (which I will present if there is still time).

Some Remarks of the Main Theorem

- It is only a categorification in the weak sense. Maps between sheaves and relations between them should be pursued in the future.

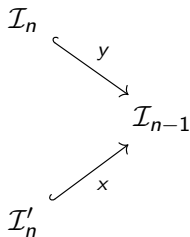
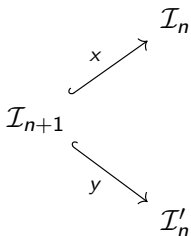
Some Remarks of the Main Theorem

- It is only a categorification in the weak sense. Maps between sheaves and relations between them should be pursued in the future.
- We could also consider the categorification of the action on higher rank stable sheaves with a refinement of the techniques.

Proof

$e_k f_l$ and $f_k e_l$ revisited

We consider the following triple/quadruple moduli space $\mathfrak{Z}_+, \mathfrak{Z}_-, \mathfrak{Y}$
parameterize diagrams:



$e_k f_l$ and $f_k e_l$ revisited

$$\begin{array}{ccc}
 & \mathcal{I}_n & \\
 \nearrow x & & \searrow y \\
 \mathcal{I}_{n+1} & & \mathcal{I}_{n-1} \\
 \searrow y & & \nearrow x \\
 & \mathcal{I}'_n &
 \end{array} \tag{7}$$

respectively, of ideal sheaves where each successive inclusion is colength 1 and supported at the point indicated on the diagrams. We consider line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}'_1, \mathcal{L}'_2$ over triple/quadruple moduli spaces with fiber $\mathcal{I}_n/\mathcal{I}_{n+1}, \mathcal{I}_{n-1}/\mathcal{I}_n, \mathcal{I}_n/\mathcal{I}'_{n+1}, \mathcal{I}'_{n-1}/\mathcal{I}_n$ respectively.

$e_k f_l$ and $f_k e_l$ revisited

The forgetful morphism induces a Cartesian diagram:

$$\begin{array}{ccc}
 \mathfrak{Y} & \xrightarrow{\alpha_+} & \mathfrak{Z}_+ \\
 \downarrow \alpha_- & \searrow \theta & \downarrow \beta_+ \\
 \mathfrak{Z}_- & \xrightarrow{\beta_-} & \mathcal{S}[n] \times \mathcal{S}[n] \times \mathcal{S} \times \mathcal{S}
 \end{array}$$

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 \end{array}$$

Lemma

$$e_k f_m = R\beta_{-*} \mathcal{L}_2^m \mathcal{L}'^{k-1}$$

$$f_m e_k = R\beta_{+*} \mathcal{L}_1^k \mathcal{L}'^{m-1}$$

Vanishing Theorem

Theorem (Vanishing Theorem)

- $R\alpha_{-*}\mathcal{O}_{\mathfrak{Z}} = \mathcal{O}_{\mathfrak{Z}_-}$
- $R\alpha_{+*}\mathcal{O}_{\mathfrak{Z}} = \mathcal{O}_{W_0}$, where W_0 and $W_1 = S^{[n,n+1]}$ are two irreducible components of \mathfrak{Z}_+ .

The proof of above theorem relies on the Minimal Model Program, and will be explained in the last part of our talk.

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Corollary

- $e_k f_m = R\theta_* \mathcal{L}_2^m \mathcal{L}_2'^{k-1}$;
- *there exists a natural transformation between $f_m e_k$ and $R\theta_*(\mathcal{L}_1^k \mathcal{L}_1'^{m-1})$*

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Corollary

- $e_k f_m = R\theta_* \mathcal{L}_2^m \mathcal{L}_2'^{k-1}$;
- *there exists a natural transformation between $f_m e_k$ and $R\theta_*(\mathcal{L}_1^k \mathcal{L}_1'^{m-1})$*

Given the above corollary, we are able to compare $e_m f_l$ and $f_l e_m$ through line bundles on \mathfrak{Y} .

Comparison Theorem

Theorem

- *The nested Hilbert scheme*
 $S^{[n-1,n,n+1]} = S^{[n-1,n]} \times_{S^{[n]}} S^{[n,n+1]}$ is a smooth divisor. of \mathfrak{H}
- We have $\mathcal{L}_1 \mathcal{L}'_2^{-1} = \mathcal{L}'_1 \mathcal{L}_2^{-1} = \mathcal{O}(S^{[n-1,n,n+1]})$

Minimal Model Program and Rational Singularities

Some Basic Ideas of the Minimal Model Program

Definition (Discrepancy)

- X be a normal variety that mK_X is Cartier for $m \in \mathbb{Z}_{>0}$

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- Suppose $f : Y \rightarrow X$ is a birational morphism from a smooth variety Y .

Remark

In birational geometry, people care more about pairs (X, D) , where D is a \mathbb{Q} or \mathbb{R} Cartier divisor. We will not present the definition of pairs for simplicity, but it is essential in part of our proof.

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Definition (Discrepancy)

- X be a normal variety that mK_X is Cartier for $m \in \mathbb{Z}_{>0}$
- Suppose $f : Y \rightarrow X$ is a birational morphism from a smooth variety Y .
- There are rational numbers $a(E_i, X)$ such that

$$\mathcal{O}_Y(mK_Y) \cong f^* \mathcal{O}_X(mK_X) \otimes \mathcal{O}_Y\left(\sum_i m a(E_i, X) E_i\right).$$

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$$\mathcal{O}_Y(mK_Y) \cong f^* \mathcal{O}_X(mK_X) \otimes \mathcal{O}_Y\left(\sum_i m a(E_i, X) E_i\right).$$

- $a(E_i, X)$ is called the discrepancy of E_i with respect to X .

Remark

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An Overview of the Minimal Model Program

Definition (Classification of singularities)

Let X be a normal variety. Assume that mK_X is Cartier for some $m > 0$. We say that X is

$$\left\{ \begin{array}{l} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{dlt} \\ \textit{lc} \end{array} \right. \quad \text{if } a(E, X) \text{ is } \left\{ \begin{array}{l} > 0, \text{ for every exceptional } E \\ \geq 0, \text{ for every exceptional } E \\ > -1, \text{ for every } E \\ > -1, \text{ for every exceptional } E \\ > -1, \text{ if } \textit{center}_X E \subset \textit{non-snc} X \\ \geq -1, \text{ for every } E \end{array} \right.$$

Remark

For S_2 and equidimensional schemes, we could also define "semi-lc", "semi-dlt".

The Structure Theorem

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- Every irreducible component of a semi-dlt scheme is normal.

By explicit computing the discrepancy, we have the following structure theorem for the geometry of \mathfrak{Y} , \mathfrak{Z}_- , \mathfrak{Z}_+

Theorem (Structure Theorem)

- \mathfrak{Y} is smooth (Negut);
- \mathfrak{Z}_+ is semi-dlt and W_0 is a canonical singularity (Y. Zhao).
- \mathfrak{Z}_- is a canonical singularity (Y. Zhao).

Review

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- For higher rank stable sheaves, \mathfrak{Z}_+ is no longer equi-dimensional, and the influence of DAG has to be accounted for.

Review

- We categorify the commutation of e_k, f_l action on the Grothendieck groups of Hilbert schemes
- For higher rank stable sheaves, \mathfrak{Z}_+ is no longer equi-dimensional, and the influence of DAG has to be accounted for.
- It is a toy model of the categorical version “Drinfeld Double of the CoHA”, and we expect the generalization to other more complicated settings.